

# A Decomposition for Hardy Martingales III

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## Abstract

We prove Davis decompositions for vector valued Hardy martingales and illustrate their use. This paper continues [17] and [18] on Davis and Garsia Inequalities.

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# 1 Introduction

The book by A. Pelczynski [19], “Banach Spaces of analytic functions and absolutely summing operators, (1977)” contains -inter alia- the following problems:

1. Does  $H^1$  have an unconditional basis?
2. Does there exist a subspace of  $L^1/H^1$  isomorphic to  $L^1$ ?
3. Does  $L^1/H^1$  have cotype 2?
4. Are the spaces  $A(\mathbb{T}^n)$  and  $A(\mathbb{T}^m)$  not isomorphic when  $n \neq m$ ?

It is well known that the solutions to those four problems were obtained by B. Maurey [16] and J. Bourgain [1, 2, 3] respectively. A common feature of the proofs by Maurey and Bourgain is the systematic use of certain complex analytic martingales. Those were studied in detail by D.J.H. Garling [11, 12] who coined the term *Hardy martingales*.

**Scalar valued Hardy martingales** were developed from different viewpoints by B. Maurey [16] and by J. Bourgain [1] to obtain the isomorphisms that gave the solution of the first two problems.

Motivated by [1] we showed recently [17], that any scalar valued Hardy martingale  $F$  may be decomposed as  $F = G + H$  into the sum of two Hardy martingales so that

$$|\Delta G_k| \leq C|F_{k-1}| \quad \text{and} \quad \mathbb{E} \sum |\Delta B_k| \leq C\|F\|_{L^1}, \quad (1.1)$$

where  $\Delta B_k = B_k - B_{k-1}$  and  $\Delta G_k = G_k - G_{k-1}$ . We used a non-linear telescoping trick to derive from (1.1) the Davis and Garsia inequalities for scalar valued Hardy martingales

$$\mathbb{E}(\sum \mathbb{E}_{k-1} |\Delta G_k|^2)^{1/2} + \mathbb{E} \sum |\Delta B_k| \leq C\|F\|_{L^1}. \quad (1.2)$$

The estimates (1.1) and (1.2) are specific for Hardy martingales and cease to hold true in general. In [18] we determined the extent to which (1.1) and (1.2) are stable under dyadic perturbations, and described the role of the perturbed estimates in the proof that  $L^1$  embeds into  $L^1/H^1$ .

**Vector valued Hardy martingales** were crucial in the solution of problems 3. and 4. The martingale inequalities that gave rise to the cotype 2 property of the quotient space  $L^1/H^1$ , and the isomorphic invariant that distinguishes between the polydisk algebras in different dimensions, are expressed in terms of vector valued Hardy martingales. See [3, 2, 4]. Bourgain’s isomorphic invariant [2] quantifies the fact that Hardy martingales ranging in the dual spaces  $A^*(\mathbb{T}^n)$  respectively  $A^*(\mathbb{T}^m)$  behave significantly different when  $m \neq n$ . The vector valued Riesz product studied by G. Pisier (see [7]) gave rise to Hardy martingales with values in  $L^1/H^1$  that intertwine the cotype 2 properties of  $L^1/H^1$ , and Bourgain’s isomorphic invariants in [2]. It also played an important role for the work of W. Davis, D. J. H. Garling, N. Tomczak-Jaegermann [7] on Hardy martingale cotype and complex uniformly convex renormings of Banach spaces.

**In the present paper** we study decompositions for vector valued Hardy martingales. Our point of reference is the following theorem of B. Davis [6]. If an  $X$  valued martingale  $F = (F_k)$  satisfies

$$\mathbb{E}(\sup_{k \in \mathbb{N}} \|F_k\|_X) < \infty,$$

then there exist martingales  $G = (G_k)$  and  $B = (B_k)$  with  $F_k = G_k + B_k$ ,  $k \in \mathbb{N}$  so that

$$\|\Delta G_k\|_X \leq C \max_{m \leq k-1} \|F_m\|_X \quad \text{and} \quad \mathbb{E} \sum \|\Delta B_k\|_X \leq C \mathbb{E}(\sup_{k \in \mathbb{N}} \|F_k\|_X), \quad (1.3)$$

The vector valued decomposition theorem of B. Davis and our previous work on scalar valued Hardy martingales, [17] and [18], gave rise to the following questions:

1. Is it still possible to prove this decomposition under the additional constraint that  $F$  and  $G, B$  are vector valued Hardy martingales?

It is easy to see, and well known, that the original proof by Davies [6, 13] does not preserve the class of Hardy martingales. In this paper we therefore use a new decomposition that respects the condition of analyticity and simultaneously yields the estimates (1.3). See Theorem 3.1. The construction is based on Brownian motion stopping times and Doob's martingale projection operator.

We apply the decomposition theorem 3.1 to prove an extrapolation result for operators acting on Hardy martingales. Theorem 3.2.

2. Is it possible to further exploit that  $F$  is taken in the class of Hardy martingale and obtain an improved decomposition with estimates that go beyond (1.3)?

In response to this question in Theorem 3.4 we obtain a decomposition of  $F$  into Hardy martingales  $F = G + B$ , for which we prove the following estimates

$$\|\Delta G_k\|_X \leq C \|F_{k-1}\|_X \quad \text{and} \quad \mathbb{E} \sum \|\Delta B_k\|_X \leq C \mathbb{E} \|F\|_X. \quad (1.4)$$

The splitting itself is done again by Brownian motion, stopping times and Doob's projection; the verification of (1.4) relies on Havin's lemma and outer functions.

The estimates (1.4) and (1.3) hold true for any complex Banach space; thus Hardy martingales are to general martingales as (1.4) is to (1.3).

With the decomposition estimates (1.4) and hypothesis " $\mathcal{H}(q)$ " we obtain further inequalities for vector valued for Hardy martingales. Let  $q \geq 2$ . A Banach space  $X$  satisfies the hypothesis  $\mathcal{H}(q)$  if for each  $M \geq 1$  there exists  $\delta = \delta(M) > 0$  such that for any  $x \in X$  with  $\|x\| = 1$  and  $g \in H_0^\infty(\mathbb{T}, X)$  with  $\|g\|_\infty \leq M$ ,

$$\int_{\mathbb{T}} \|z + g\|_X dm \geq (1 + \delta (\int_{\mathbb{T}} \|g\|_X^q dm)^q)^{1/q}. \quad (1.5)$$

Theorem 3.7 asserts that if the Banach space  $X$  satisfies  $\mathcal{H}(q)$  then any  $X$ -valued Hardy martingale  $F$  has a decomposition into Hardy martingales as  $F = G + B$  such that

$$\mathbb{E} \left( \sum_{k=1}^{\infty} (\mathbb{E}_{k-1} \|\Delta G_k\|_X^q) \right)^{1/q} + \mathbb{E} \left( \sum_{k=1}^{\infty} \|\Delta B_k\|_X \right) \leq A_q \mathbb{E}(\|F\|_X).$$

If we replace (1.5) by the weaker hypothesis

$$\int_{\mathbb{T}} \|z + g\|_X dm \geq (1 + \delta(\int_{\mathbb{T}} \|g\|_X dm)^q)^{1/q}, \quad (1.6)$$

then we are able to prove that the decomposition estimates (1.4) yield

$$\mathbb{E}(\sum_{k=1}^{\infty} (\mathbb{E}_{k-1} \|\Delta G_k\|_X)^q)^{1/q} + \mathbb{E}(\sum_{k=1}^{\infty} \|\Delta B_k\|_X) \leq A_q \mathbb{E}(\|F\|_X).$$

We note in passing that for scalar valued analytic functions, when  $X = \mathbb{C}$ , the conditions (1.5) and (1.6) hold true with  $q = 2$ . See [1, 17].

3. Illustrating the use of Brownian Motion we give a simple proof of the fact that any Hardy martingale can be embedded -as a subsequence- into another Hardy martingale with small and previsible increments. Theorem 4.2 should probably be regarded as a weak version of Q. Xu's embedding theorem referred to by Garling [11]. See also the construction of G. Edgar [9, 10].

## 2 Preliminaries

**Hardy spaces.** Let  $X$  be a complex Banach space. For  $1 \leq p \leq \infty$  we denote by  $L_0^p(\mathbb{T}, X)$ , the Bochner space of  $X$  valued  $p$ -integrable, functions with vanishing mean. Here  $\mathbb{T} = \{e^{i\theta} : \theta \in [0, 2\pi]\}$  is the torus equipped with the normalized angular measure. We define  $H_0^p(\mathbb{T}, X) \subset L_0^p(\mathbb{T}, X)$  to consist of those functions for which the harmonic extension to the unit disk is analytic. See [20], [9], [11].

**Martingales on  $\mathbb{T}^{\mathbb{N}}$ .** Let  $\mathbb{T}^{\mathbb{N}}$  be the countable torus product equipped with its normalized Haar measure  $\mathbb{P}$ . We enote by  $\mathcal{F}_n$  the sigma-algebra on  $\mathbb{T}^{\mathbb{N}}$  generated  $\{(A_1, \dots, A_n, \mathbb{T}^{\mathbb{N}})\}$ , where  $A_i, i \leq n$  are measurable subsets of  $\mathbb{T}$ . Let  $F = (F_n)$  be a sequence in the Bochner space  $L^1(\mathbb{T}^{\mathbb{N}}, X)$ — so that  $F_n$  is  $\mathcal{F}_n$  measurable. It is an  $(\mathcal{F}_n)$  martingale if conditioned on  $\mathcal{F}_{n-1}$  the difference  $\Delta F_n = F_n - F_{n-1}$  defines an element in  $L_0^1(\mathbb{T}, X)$ . Doob's maximal function estimate states that

$$(\mathbb{E} \sup_{k \in \mathbb{N}} \|F_k\|_X^p)^{1/p} \leq \frac{p}{p-1} (\sup_{k \in \mathbb{N}} \mathbb{E} \|F_k\|_X^p)^{1/p}, \quad 1 < p \leq \infty, \quad (2.1)$$

for every  $X$  valued  $(\mathcal{F}_n)$  martingale. ( See e.g. [8]. )

Assume now that  $F = (F_k)$  is an  $X$  valued  $(\mathcal{F}_n)$  martingale. It is called a *Hardy martingale* if conditioned to  $\mathcal{F}_{n-1}$ , the martingale difference  $\Delta F_n = F_n - F_{n-1}$  defines an element in  $H_0^1(\mathbb{T}, X)$ . See Garling [11], Pisier [20].

**Brownian motion.** Let  $\Omega$  denote the Wiener space. We let  $\{z_t : t > 0\}$  denote complex Brownian motion started at  $0 \in \mathbb{D}$ , let  $\{\mathcal{F}_t : t > 0\}$  denote its associated continuous filtration, and define the stopping time  $\tau$  to denote the first time when Brownian motion  $\{z_t : t > 0\}$  hits the boundary of the unit disk, thus

$$\tau = \inf\{t > 0 : |z_t| > 1\}.$$

We recall that for any  $f \in H^1(\mathbb{T}, X)$  and  $0 < \alpha \leq 1$ ,  $(\|f(z_{t \wedge \tau})\|_X^\alpha)$  is a submartingale, and that Garling's inequality [11] asserts that

$$\mathbb{E}(\sup_{t < \tau} \|f(z_t)\|_X) \leq e \sup_{t < \tau} \mathbb{E}(\|f(z_t)\|_X),$$

where the integration is taken over the Wiener space  $\Omega$ . We recall *Doob's projection operator*  $N : L^p(\Omega, X) \rightarrow L^p(\mathbb{T}, X)$  acting, by conditional expectation, on random variables defined on Wiener space  $\Omega$ ,

$$Nf(z) = \mathbb{E}(f|z_\tau = z), \quad z \in \mathbb{T}.$$

We use Doob's martingale projection to generate analytic functions in  $H^\infty(\mathbb{T}, X)$  by the following stopping time procedure. For  $f \in H^1(\mathbb{T}, X)$ ,  $\lambda > 0$  put

$$\rho = \inf\{t < \tau : \|f(z_t)\|_X > \lambda\}, \quad R = f(z_\rho), \quad g = N(R).$$

Then  $g$  is analytic and uniformly bounded by  $\lambda$ . Precisely,

$$\|g\|_X \leq \lambda, \quad g \in H^\infty(\mathbb{T}, X). \quad (2.2)$$

See [23] for the original argument based on duality, and [14] for an alternative proof, based on Ito calculus.

**Maximal function estimates for Hardy martingales [11].** Let  $X$  be a Banach space and let  $F = (F_k)$  be an integrable  $X$  valued Hardy martingale. For any  $0 < \alpha \leq 1$ ,  $(\|F_k\|_X^\alpha)$  is a non-negative submartingale,

$$\|F_{k-1}\|_X^\alpha \leq \mathbb{E}_{k-1}(\|F_k\|_X^\alpha),$$

and

$$\mathbb{E}(\sup_{k \in \mathbb{N}} \|F_k\|_X) \leq e \sup_{k \in \mathbb{N}} \mathbb{E}(\|F_k\|_X). \quad (2.3)$$

Moreover for any  $k \in \mathbb{N}$ , Garling's theorem [11] yields that the Brownian maximal function

$$F_k^*(x, \omega) = \max \left\{ \max_{m \leq k-1} \|F_m(x)\|_X, \sup_{t < \tau} \|F_k(x, z_t(\omega))\|_X \right\}, \quad x \in \mathbb{T}^{k-1}$$

is integrable over  $\Sigma = \mathbb{T}^{k-1} \times \Omega$  and

$$\mathbb{E}_\Sigma(F_k^*) \leq C \mathbb{E}(\|F_k\|_X). \quad (2.4)$$

## 3 Vector Valued Hardy Martingale Decompositions

### 3.1 The classical Davis decomposition

Here we present martingale decompositions that preserve the class of vector valued Hardy martingales. We split such an  $F$  as  $F = G + B$  where  $G$  is a vector valued Hardy martingale with predictable increments, and where the martingale differences of  $B$  are absolutely summing. The proof combines Davis's original idea and maximal function estimates (2.3), (2.4) and the fact that Doob's projection  $N$  preserves analyticity (2.2).

**Theorem 3.1.** *Let  $X$  be a Banach space. Any  $X$  valued Hardy martingale  $F = (F_k)_{k=1}^n$  can be decomposed into the sum of Hardy martingales  $F = G + B$  such that*

$$\|\Delta G_k\|_X \leq 2 \max_{m \leq k-1} \|F_m\|_X,$$

and

$$\mathbb{E}\left(\sum_{k=1}^n \|\Delta B_k\|_X\right) \leq C\mathbb{E}(\|F\|_X).$$

PROOF. Fix  $k \in \mathbb{N}$  and condition to  $\mathcal{F}_{k-1}$ . That is we fix  $x \in \mathbb{T}^{k-1}$ ,  $v \in \mathbb{T}$  and put

$$f(v) = \Delta F_k(x, v), \quad \lambda = \max_{m \leq k-1} \|F_m(x)\|_X.$$

Define

$$\rho = \inf\{t < \tau : \|f(z_t)\|_X > 2\lambda\}, \quad A = \{\rho < \tau\}.$$

Now put

$$R_k = f(z_\rho), \quad S_k = f(z_\rho) - f(z_\tau).$$

We next analyse the properties of  $R_k$ ,  $G_k$ . For  $\omega \in A$ ,

$$F_k^*(x, \omega) \leq 4(F_k^*(x, \omega) - F_{k-1}^*(x, \omega)).$$

and by definition  $S_k$  is supported on  $A$ , hence  $\|S_k\|_X \leq 2F_k^* \leq 8(F_k^* - F_{k-1}^*)$ , and

$$\sum_{k=1}^n \|S_k\|_X \leq 8F_n^*. \quad (3.1)$$

On the other hand, by choice of the stopping times  $\rho$ , we have

$$\|R_k\| \leq 2\lambda, \quad (3.2)$$

Use Doob's martingale projection to generate analytic functions. Define

$$\Delta G_k = N(R_k), \quad \Delta B_k = N(S_k),$$

where  $N$  acts on the last variable of  $S_k, R_k$ . Clearly  $\Delta F_k = \Delta G_k + \Delta B_k$ , and since Doob's projection preserves analyticity,  $(G_k)$  and  $(B_k)$  form Hardy martingales. By convexity, the interpretation of Doob's projection  $N$  as a conditional expectation, together with (3.1), and (2.4) gives

$$\mathbb{E}\left(\sum_{k=1}^n \|\Delta B_k\|_X\right) \leq \mathbb{E}\left(\sum_{k=1}^n \|S_k\|_X\right) \leq 8\mathbb{E}(F_n^*) \leq C\mathbb{E}(\|F_n\|_X).$$

Using once again that Doob's projection  $N$  acts as a conditional expectation operator, we get with (3.2)

$$\|\Delta G_k\|_X = \|N(R_k)\|_X \leq 2 \max_{m \leq k-1} \|F_m(x)\|_X$$

■

### 3.2 Illustration: Extrapolation of Hardy-Martingale Transforms

Throughout this section we fix a Banach space  $X$  and  $\varepsilon = (\varepsilon_m) \in \{-1, 1\}^{\mathbb{N}}$ . We define the operators

$$T_\varepsilon(F) = \sum \varepsilon_m \Delta F_m, \quad T_\varepsilon(F)_k = \sum_{m=1}^k \varepsilon_m \Delta F_m. \quad (3.3)$$

initially for finite,  $X$  valued Hardy martingales  $F = (F_k)_{k=1}^n$ .

Illustrating how the Davis decomposition for vector valued Hardy martingales may be applied, we combine it with an extrapolation method for *previsible* martingales (Maurey [15].) Thus Theorem 3.1 yields Garling's [11] extrapolation theorem.

**Theorem 3.2.** *If there exists  $A_2 > 0$  such that for any square integrable,  $X$  valued Hardy martingale  $Z = (Z_k)$*

$$\mathbb{E}(\|T_\varepsilon(Z)_k\|_X^2) \leq A_2^2 \mathbb{E}(\|Z_k\|_X^2), \quad k \in \mathbb{N},$$

*then there exists  $A_1 = A_1(A_2)$  such that for any integrable  $X$  valued Hardy martingale  $F = (F_k)$*

$$\mathbb{E}(\|T_\varepsilon(F)_k\|_X) \leq A_1 \mathbb{E}(\|F_k\|_X), \quad k \in \mathbb{N}.$$

**Remarks:** The proof by Garling [11] combined extrapolation for previsible martingales (e.g. Burkholder [5]) and used that Q. Xu has shown that Edgar's approximation argument ([9], [10]) reduces the problem to a special case, called analytic martingales. For a recent study of the operators  $T_\varepsilon$  we refer to the results in the thesis of Yanqi Qiu [22, 21].

We first recall Maurey's extrapolation argument [15].

**Lemma 3.3.** *(Maurey [15].) Assume that there exists  $A_2 > 0$  so that for any square integrable,  $X$  valued Hardy martingale  $Z = (Z_k)$*

$$\mathbb{E}(\|T_\varepsilon(Z)_k\|_X^2) \leq A_2^2 \mathbb{E}(\|Z_k\|_X^2) \quad k \in \mathbb{N}.$$

*Let  $G = (G_k)$  be an  $X$  valued integrable Hardy martingale. Let  $w = (w_k)$  be a non negative, increasing and adapted sequence satisfying*

$$\max_{m \leq k} \|G_m\|_X \leq w_{k-1}. \quad (3.4)$$

*Then*

$$\mathbb{E}(\|T_\varepsilon(G)_k\|_X) \leq 8A_2 \mathbb{E}(w_{k-1}), \quad k \in \mathbb{N}.$$

PROOF. We follow the basic steps of Maurey's argument in [15].

**Step 1.** Given  $G = (G_k)$  define the transformed Hardy martingale

$$Z_k = \sum_{m=1}^k w_{m-1}^{-1/2} \Delta G_m, \quad k \in \mathbb{N}.$$

**Step 2.** We infer from Maurey [15] that with (3.4), the transformed Hardy martingale  $Z = (Z_k)$  satisfies the pointwise estimates

$$\|Z_k\|_X \leq 2w_{k-1}^{1/2}, \quad (3.5)$$

and

$$\|T_\varepsilon(G)_k\|_X \leq 2(\max_{m \leq k} \|T_\varepsilon(Z)_m\|_X)w_{k-1}^{1/2}. \quad (3.6)$$

**Step 3.** By (3.6), the Cauchy Schwarz inequality and Doob's maximal theorem we obtain

$$\mathbb{E}(\|T_\varepsilon(G)_k\|_X) \leq 2(\mathbb{E}\|\max_{m \leq k} T_\varepsilon(Z)_m\|_X^2)^{1/2}(\mathbb{E}(w_{k-1}))^{1/2} \leq 4(\mathbb{E}\|T_\varepsilon(Z)_k\|_X^2)^{1/2}(\mathbb{E}(w_{k-1}))^{1/2}.$$

Next, by the hypothesis on  $T_\varepsilon$  and the pointwise bound (3.5), we get

$$\mathbb{E}(\|T_\varepsilon(Z)_k\|_X^2) \leq A_2^2 \mathbb{E}(\|Z_k\|_X^2) \leq 4A_2^2 \mathbb{E}(w_{k-1}).$$

Summing up we have

$$\mathbb{E}(\|T_\varepsilon(G)_k\|_X) \leq 8A_2 \mathbb{E}(w_{k-1}).$$

■

**Proof of Theorem 3.2.** With Theorem 3.1 decompose the Hardy martingale as  $F = G + H$ . We use Lemma 3.3 to estimate  $T_\varepsilon(G)$  and the triangle inequality for  $T_\varepsilon(B)$ .

**Step 1.** Apply Theorem 3.1 to the  $X$  valued Hardy martingale  $F = (F_k)$  and obtain the splitting as  $F = G + H$  such that

$$\|\Delta G_k\|_X \leq 2 \max_{m \leq k-1} \|F_m\|_X, \quad \text{and} \quad \mathbb{E}\left(\sum_{m=1}^k \|\Delta B_m\|_X\right) \leq C_0 \mathbb{E}(\|F_k\|_X), \quad (3.7)$$

where again  $G, B$  are  $X$  valued Hardy martingales.

**Step 2.** Put

$$w_{k-1} = 2 \max_{m \leq k-1} \|F_m\|_X + \max_{m \leq k-1} \|G_m\|_X.$$

By (3.7),  $\|G_k\|_X \leq w_{k-1}$ . Hence Lemma 3.3 applies and gives

$$\mathbb{E}(\|T_\varepsilon(G)_k\|_X) \leq 8A_2 \mathbb{E}(w_{k-1}). \quad (3.8)$$

By (3.7), and the maximal function estimates for Hardy martingales in (2.3) we have

$$\mathbb{E}(w_{k-1}) \leq C_1 \mathbb{E}(\|F_k\|_X). \quad (3.9)$$

**Step 3.** Next we turn to estimating  $T_\varepsilon(B)_k$ . We use (3.7) and triangle inequality as follows,

$$\mathbb{E}(\|T_\varepsilon(B)_k\|_X) \leq \mathbb{E}\left(\sum_{m=1}^k \|\Delta B_m\|_X\right) \leq C_0 \mathbb{E}(\|F_k\|_X). \quad (3.10)$$

Summing up the estimates (3.8) – (3.10) we get

$$\mathbb{E}(\|T_\varepsilon(F)_k\|_X) \leq \mathbb{E}(\|T_\varepsilon(G)_k\|_X) + \mathbb{E}(\|T_\varepsilon(B)_k\|_X) \leq (8A_2C_1 + C_0)\mathbb{E}(\|F_k\|_X).$$

■



### 3.3 The Strong Davis Decomposition

We continue with decomposition theorems. In Theorem 3.4 we determine a splitting of a vector valued Hardy martingale  $F$  as  $F = G + B$  that improves upon the classical Davis decomposition of Theorem 3.1. Specifically the uniform estimates for the predictable part  $G$  are improved.

In addition to Brownian motion and stopping times, the proof below makes use of Havin's Lemma for which we refer to A. Pelczynski [19] and J. Bourgain [1].

**Theorem 3.4.** *Let  $X$  be a Banach space. Any  $X$  valued Hardy martingale  $F = (F_k)$  can be decomposed into the sum of  $X$  valued Hardy martingales  $F = G + B$  such that*

$$\|\Delta G_k\|_X \leq C\|F_{k-1}\|_X,$$

and

$$\mathbb{E}\left(\sum_{k=1}^{\infty} \|\Delta B_k\|_X\right) \leq C\mathbb{E}(\|F\|_X).$$

The splitting of the Hardy martingale  $F$  is done separately for each martingale difference. Here the proof relies on the following decomposition theorem for vector valued analytic functions.

**Theorem 3.5.** *For any  $h \in H_0^1(\mathbb{T}, X)$  and  $z \in X$  there exists  $g \in H_0^\infty(\mathbb{T}, X)$  so that*

$$\|g(\zeta)\|_X \leq C_0\|z\|_X, \quad \zeta \in \mathbb{T}$$

and

$$\|z\|_X + \frac{1}{8} \int_{\mathbb{T}} \|h - g\|_X dm \leq \int_{\mathbb{T}} \|z + h\|_X dm. \quad (3.11)$$

The constant satisfies  $C_0 \leq 24$ .

PROOF. The proof begins with the definition of  $g \in H_0^\infty(\mathbb{T}, X)$ . Thereafter we successively collect the lower estimates for the right hand side of (3.11).

**Step 1.** Determine  $g \in H_0^\infty(\mathbb{T}, X)$  by putting

$$\rho = \inf\{t < \tau : \|h(z_t)\|_X > C_0\|z\|_X\}, \quad g = N(h(z_\rho)),$$

where  $N$  denotes Doob's projection operator. Since  $h \in H_0^1(\mathbb{T}, X)$  we have  $\mathbb{E}(h(z_\rho)) = 0$ . By definition of the stopping time  $\rho$  we have the uniform estimate  $\|h(z_\rho)\|_X \leq C_0\|z\|_X$  and we obtain

$$\|g(\zeta)\|_X \leq C_0\|z\|_X, \quad \zeta \in \mathbb{T},$$

because Doob's projection  $N$  being a conditional expectation, is a contraction between  $L^\infty$  spaces. Finally since  $N$  preserves analyticity, we get  $g \in H_0^\infty(\mathbb{T}, X)$ .

**Step 2.** We now turn to proving the integral estimates. The idea is to find a lower estimate for the right hand side by integrating it against a suitable testing functions. Define  $A = \{\rho < \tau\}$ . The set  $A$  is measurable with respect to the stopping time  $\sigma$ -algebra  $\mathcal{F}_\rho$ . Since conditional expectations are  $L^1$  contractions we have

$$\mathbb{E}(\|h(z_\tau)1_A\|_X) \geq \mathbb{E}(\|h(z_\rho)1_A\|_X) \geq C_0\|z\|_X. \quad (3.12)$$

Next define

$$p = N(1_A)/2.$$

Clearly  $0 \leq p \leq 1/2$  and by the covariance formula we get  $\int p dm = \mathbb{P}(A)/2$ , and

$$\int_{\mathbb{T}} \|h\|_X p dm = \frac{1}{2} \mathbb{E}(\|h(z_\tau)1_A\|_X).$$

Combining this with (3.12) triangle inequality gives

$$\int_{\mathbb{T}} \|z + h\|_X p dm \geq \int_{\mathbb{T}} \|h\|_X p dm - \frac{1}{2} \mathbb{P}(A) \|z\|_X \geq \left(\frac{1}{2} - \frac{1}{2C_0}\right) \mathbb{E}(\|h(z_\tau)1_A\|_X). \quad (3.13)$$

**Step 3.** Let  $q \in H^\infty(\mathbb{T})$  be the outer function given by

$$q = \exp(\ln(1 - p) + iH \ln(1 - p)).$$

Since  $h \in H_0^1(\mathbb{T}, X)$  we have  $\int_{\mathbb{T}} h q dm = 0$ . Put  $q_2 = \Im q$  and  $q_1 = \Re q$ . Then by inspection  $\int q_2 dm = 0$  and

$$\int_{\mathbb{T}} (z + h) q dm = z \int_{\mathbb{T}} q_1 dm.$$

Below we will verify that for  $C_1 > 3$

$$\int_{\mathbb{T}} q_1 dm > 1 - C_1 \mathbb{P}(A). \quad (3.14)$$

Assuming the crucial estimate (3.14) for the moment we may continue our chain of inequalities as follows.

$$\int_{\mathbb{T}} \|z + h\|_X |q| dm \geq \left\| \int_{\mathbb{T}} (z + h) q dm \right\|_X \geq \|z\|_X (1 - C_1 \mathbb{P}(A)). \quad (3.15)$$

Finally we observe that  $p + |q| = 1$  and take the sum of (3.13) and (3.15) to obtain

$$\int_{\mathbb{T}} \|z + h\|_X dm \geq \|z\|_X + \left(\frac{1}{2} - \frac{1}{2C_0} - \frac{C_1}{C_0}\right) \mathbb{E}(\|h(z_\tau)1_A\|_X). \quad (3.16)$$

**Step 4.** Here we prove that

$$\int_{\mathbb{T}} \|h - g\|_X \leq 2 \mathbb{E}(\|h(z_\tau)1_A\|_X). \quad (3.17)$$

As  $A = \{\rho < \tau\}$ , the following identity holds

$$h(z_\tau) - h(z_\rho) = (h(z_\tau) - h(z_\rho))1_A. \quad (3.18)$$

Using that Doob's projection contracts  $L^1$  spaces, we derive from (3.18) that

$$\int_{\mathbb{T}} \|h - g\|_X dm \leq \mathbb{E}(\|h(z_\tau) - h(z_\rho)\|_X 1_A).$$

Next use the right hand inequality in (3.12) to get

$$\mathbb{E}(\|h(z_\tau) - h(z_\rho)\|_X 1_A) \leq 2\mathbb{E}(\|h(z_\tau) 1_A\|_X).$$

**Summing up.** Choose now  $C_0 \geq 8C_1$  so that  $(1/2 - 1/(2C_0) - C_1/C_0) > 1/4$  and merge the inequalities (3.16) – (3.17) to obtain

$$\int_{\mathbb{T}} \|z + h\|_X dm \geq \|z\|_X + \frac{1}{8} \int_{\mathbb{T}} \|h - g\|_X dm.$$

■

**A final remark.** Here we isolate Bourgain's idea [1] used to prove that  $0 \leq p \leq 1/2$  implies that (3.14) holds true. We show

$$\int_{\mathbb{T}} q_1 dm > 1 - 3 \int_{\mathbb{T}} p dm. \quad (3.19)$$

Recall that  $q_1 = (1 - p) \cos(H(\ln(1 - p)))$ . Use  $\cos(x) \geq 1 - x^2/2$  to get the pointwise inequality

$$q_1 = (1 - p) - \frac{1}{2}(H((1 - p)))^2. \quad (3.20)$$

We thus reduced the  $L^1$  estimate for  $q_1$  to an  $L^2$  estimate for the Hilbert transform. Clearly we have

$$\int_{\mathbb{T}} (H(\ln(1 - p)))^2 dm \leq 2 \int_{\mathbb{T}} (\ln(1 - p))^2 dm.$$

Now if  $0 \leq p \leq 1/2$  then  $(\ln(1 - p))^2 \leq 2p$ , and hence

$$\int_{\mathbb{T}} (H(\ln(1 - p)))^2 dm \leq 4 \int_{\mathbb{T}} p dm. \quad (3.21)$$

Combining now (3.20) and (3.21) gives (3.19).

■

**Proof of Theorem 3.4.** Let  $k \in \mathbb{N}$  and condition on  $\mathcal{F}_{k-1}$  by fixing  $x \in \mathbb{T}^{k-1}$ . For  $y \in \mathbb{T}$  put

$$h(y) = \Delta F_k(x, y) \quad \text{and} \quad z = F_{k-1}(x).$$

We apply Theorem 3.5 to  $h \in H_0^1(\mathbb{T}, X)$  and obtain  $g \in H_0^\infty(\mathbb{T}, X)$ , such that

$$\|g(\zeta)\|_X \leq C_0 \|z\|_X, \quad \zeta \in \mathbb{T}$$

and

$$\|z\|_X + \frac{1}{8} \int_{\mathbb{T}} \|h - g\|_X dm \leq \int_{\mathbb{T}} \|z + h\|_X dm.$$

Define the splitting of  $\Delta F_k$  by putting

$$\Delta G_k(x, y) = g(y), \quad \text{and} \quad \Delta B_k(x, y) = h(y) - g(y).$$

This gives  $\Delta F_k = \Delta G_k + \Delta B_k$ , with  $\|\Delta G_k\|_X \leq C_0 \|F_{k-1}\|_X$ . and

$$\|F_{k-1}\|_X + \frac{1}{8} \mathbb{E}_{k-1}(\|\Delta B_k\|_X) \leq \mathbb{E}_{k-1}(\|F_k\|_X).$$

Taking expectations on both sides and summing the telescoping series gives

$$\sum \mathbb{E}(\|\Delta B_k\|_X) \leq 8 \sup \mathbb{E}(\|F_k\|_X).$$

■

### 3.4 Illustration: Vector valued Davis and Garsia Inequality

Here we show that the strong Davis decomposition yields vector valued Davis and Garsia Inequalities. At this point we need to make an assumption on the Banach space  $X$ : Let  $q \geq 2$ . A Banach space  $X$  satisfies the hypothesis  $\mathcal{H}(q)$ , if for each  $M \geq 1$  there exists  $\delta = \delta(M) > 0$  such that for any  $x \in X$  with  $\|x\| = 1$  and  $g \in H_0^\infty(\mathbb{T}, X)$  with  $\|g\|_\infty \leq M$ ,

$$\int_{\mathbb{T}} \|z + g\|_X dm \geq (1 + \delta \int_{\mathbb{T}} \|g\|_X^q dm)^{1/q}. \quad (3.22)$$

We emphasize that (3.22) is required to hold only for uniformly bounded analytic functions  $g$ , and that  $\delta = \delta(M) > 0$  is allowed to depend on the uniform estimates  $\|g\|_\infty \leq M$ .

**Theorem 3.6.** *Let  $q \geq 2$ . Let  $X$  be a Banach satisfying  $\mathcal{H}(q)$ . There exists  $M > 0$   $\delta_q > 0$  such that for any  $h \in H_0^1(\mathbb{T}, X)$  and  $z \in X$  there exists  $g \in H_0^\infty(\mathbb{T}, X)$  satisfying*

$$\|g(\zeta)\|_X \leq M \|z\|_X, \quad \zeta \in \mathbb{T}, \quad (3.23)$$

and

$$\int_{\mathbb{T}} \|z + h\|_X dm \geq \left( \|z\|_X^q + \delta_q \int_{\mathbb{T}} \|g\|_X^q dm \right)^{1/q} + \frac{1}{16} \int_{\mathbb{T}} \|h - g\|_X dm. \quad (3.24)$$

PROOF. Let  $h \in H_0^1(\mathbb{T}, X)$  and  $z \in X$ . By Theorem 3.5 there exists  $M \leq 24$  and  $g \in H_0^\infty(\mathbb{T}, X)$  so that

$$\|g(\zeta)\|_X \leq M \|z\|_X, \quad \zeta \in \mathbb{T}, \quad (3.25)$$

and

$$\int_{\mathbb{T}} \|z + h\|_X dm \geq \|z\|_X + \frac{1}{8} \int_{\mathbb{T}} \|h - g\|_X dm. \quad (3.26)$$

Next by triangle inequality,

$$\int_{\mathbb{T}} \|z + h\|_X dm \geq \int_{\mathbb{T}} \|z + g\|_X dm - \int_{\mathbb{T}} \|h - g\|_X dm. \quad (3.27)$$

and by hypothesis  $\mathcal{H}(q)$  there exists  $\delta = \delta(M) > 0$  such that

$$\int_{\mathbb{T}} \|z + g\|_X dm \geq (\|z\|_X^q + \delta \int_{\mathbb{T}} \|g\|_X^q dm)^{1/q}. \quad (3.28)$$

Let  $0 < \alpha \leq 1$  form  $\alpha(3.26) + (1 - \alpha)(3.27)$  and invoke (3.28). Thus we obtained that  $\int \|z + h\|$  is larger than the following term,

$$(1 - \alpha)\|z\|_X + \alpha(\|z\|_X^q + \delta \int_{\mathbb{T}} \|g\|_X^q dm)^{1/q} + \frac{(1 - 9\alpha)}{8} \int_{\mathbb{T}} \|h - g\|_X dm. \quad (3.29)$$

Just by triangle inequality (3.29) is larger than

$$(\|z\|_X^q + \delta \alpha^q \int_{\mathbb{T}} \|g\|_X^q dm)^{1/q} + \frac{(1 - 9\alpha)}{8} \int_{\mathbb{T}} \|h - g\|_X dm.$$

Specifying  $\alpha = 1/18$  finishes the proof of (3.24). ■

**Theorem 3.7.** *Let  $q \geq 2$ . Let  $X$  be a Banach satisfying  $\mathcal{H}(q)$ . Any  $X$  valued Hardy martingale  $F = (F_k)$  can be decomposed into the sum of  $X$  valued Hardy martingales  $F = G + B$  such that*

$$\mathbb{E}\left(\sum_{k=1}^{\infty} \mathbb{E}_{k-1}(\|\Delta G_k\|_X^q)\right)^{1/q} + \mathbb{E}\left(\sum_{k=1}^{\infty} \|\Delta B_k\|_X\right) \leq A_q \mathbb{E}(\|F\|_X).$$

PROOF. Let  $k \in \mathbb{N}$  and condition on  $\mathcal{F}_{k-1}$ . Fix  $x \in \mathbb{T}^{k-1}$ . and  $y \in \mathbb{T}$  and define

$$h(y) = \Delta F_k(x, y) \quad \text{and} \quad z = F_{k-1}(x).$$

We apply Theorem 3.6 to  $h \in H_0^1(\mathbb{T}, X)$  and obtain  $g \in H_0^\infty(\mathbb{T}, X)$ , satisfying (3.24). Substituting back we obtain the decomposing

$$\Delta G_k(x, y) = g(y), \quad \text{and} \quad \Delta B_k(x, y) = h(y) - g(y)$$

such that

$$\mathbb{E}(\|F_{k-1}\|_X^q + \delta \mathbb{E}_{k-1}(\|\Delta G_k\|_X^q))^{1/q} + C \mathbb{E}(\|\Delta B_k\|_X) \leq \mathbb{E}(\|F_k\|_X). \quad (3.30)$$

Apply non-linear telescoping [1, 17], to equation (3.30). This gives

$$\mathbb{E}\left(\sum_{k=1}^{\infty} \mathbb{E}_{k-1}(\|\Delta G_k\|_X^q)\right)^{1/q} + \mathbb{E}\left(\sum_{k=1}^{\infty} \|\Delta B_k\|_X\right) \leq A_q (\mathbb{E}(\|F\|_X))^{1/q} (\mathbb{E}(\sup_{n \in \mathbb{N}} \|F_n\|_X))^{1/p},$$

where  $1/p + 1/q = 1$ . Invoking (2.3) –the maximal function estimate for Hardy martingales –finishes the proof. ■

**Remark:** If in the definition of  $\mathcal{H}(q)$  we had replaced (3.22) by

$$\int_{\mathbb{T}} \|z + g\|_X dm \geq (1 + \delta(\int_{\mathbb{T}} \|g\|_X dm)^q)^{1/q}, \quad (3.31)$$

then the above line of reasoning would have resulted in the previsible projection estimate

$$\mathbb{E}\left(\sum_{k=1}^{\infty} (\mathbb{E}_{k-1} \|\Delta G_k\|_X^q)^{1/q}\right) + \mathbb{E}\left(\sum_{k=1}^{\infty} \|\Delta B_k\|_X\right) \leq A_q \mathbb{E}(\|F\|_X).$$

## 4 Embedding: An Alternative to Decomposing.

Our starting point in this section is Maurey's embedding of  $H^1(\mathbb{T}, X)$  into Hardy martingales with uniformly small increments. See [16]. By iterating Maurey's construction we show that an arbitrary Hardy martingale may be considered as a subsequence of a Hardy martingale with the additional property that its increments are dominated by a small, predictable and increasing process. Our interest in this result comes from extrapolation theorems such as Burkholder's [5] or Maurey's [15]. As stated above our Theorem 4.2 is probably a weaker version of the embedding theorem of Q. Xu, referred to by Garling [11]. Nevertheless with respect to extrapolation Theorem 4.2 allows us to draw similar conclusions.

Let  $1/2 > \epsilon > 0$  and  $w \in \mathbb{T}^{\mathbb{N}}$  with  $w = (w_k)$ . We define inductively  $\varphi_1(w) = \epsilon w_1$ , and

$$\varphi_n(w) = \varphi_{n-1}(w) + \epsilon(1 - |\varphi_{n-1}(w)|)^2 w_n. \quad (4.1)$$

As proved by Maurey [16]  $\varphi = (\varphi_n)$  is a uniformly bounded Hardy martingale whose limit is uniformly distributed over  $\mathbb{T}$ , that is

$$\mathbb{P}(\{w \in \mathbb{T}^{\mathbb{N}} : \varphi(w) \in B\}) = m(B) \quad B \subseteq \mathbb{T},$$

where  $m(B)$  denotes the mormalized Haar measure on  $\mathbb{T}$ .

The following is Maurey's embedding theorem [16].

**Theorem 4.1.** *For any  $f \in H^1(\mathbb{T}, X)$*

$$F_n(w) = f(\varphi_n(w)), \quad w \in \mathbb{T}^{\mathbb{N}}$$

*defines an  $X$  valued Hardy martingale for which*

$$\sup_{n \in \mathbb{N}} \mathbb{E}(\|F_n\|_X) = \int_{\mathbb{T}} \|f\|_X dm \quad (4.2)$$

*and*

$$\|\Delta F_n\|_X \leq 2\epsilon \int_{\mathbb{T}} \|f\|_X dm. \quad (4.3)$$

PROOF. For convenience we sketch Maurey's proof. It is straightforward to see that  $(F_n)$  is indeed an integrable  $X$  valued Hardy martingale and that (4.2) holds true. We now turn to the pointwise estimates (4.3). Fix  $w \in \mathbb{T}^{\mathbb{N}}$ , and  $n \in \mathbb{N}$ . Then

$$\Delta F_n(w) = f(\varphi_n(w)) - f(\varphi_{n-1}(w)).$$

Put next  $z = \varphi_n(w)$ ,  $u = \varphi_{n-1}(w)$  and use the Cauchy integral formula to obtain

$$f(z) - f(u) = \int_{\mathbb{T}} \left\{ \frac{\zeta}{\zeta - z} - \frac{\zeta}{\zeta - u} \right\} f(\zeta) dm(\zeta).$$

By the triangle inequality we get

$$\|f(z) - f(u)\|_X \leq \frac{|z - u|}{(1 - |u|)(1 - |z|)} \int_{\mathbb{T}} \|f\|_X dm \quad (4.4)$$

We use the defining recursion (4.1) to see that

$$|\varphi_n(w) - \varphi_{n-1}(w)| = \epsilon(1 - |\varphi_{n-1}(w)|)^2 \quad (4.5)$$

and

$$(1 - |\varphi_{n-1}(w)|) \leq (1 - \epsilon)^{-1}(1 - |\varphi_n(w)|). \quad (4.6)$$

Since we put  $z = \varphi_n(w)$  and  $u = \varphi_{n-1}(w)$ , the relations (4.5) and (4.6) imply that

$$\frac{|z - u|}{(1 - |u|)(1 - |z|)} \leq 2\epsilon. \quad (4.7)$$

Combining the estimates (4.4) and (4.7) yields the following pointwise bounds for the martingale differences

$$\|\Delta F_n\|_X \leq 2\epsilon \int_{\mathbb{T}} \|f\|_X dm.$$

■

Applying Maurey's Theorem 4.1 repeatedly we associate to an arbitrary Hardy martingale a subsequence of a Hardy martingale *with small predictable increments* and almost identical norms. As mentioned above this makes it possible to apply standard extrapolation theorems *without performing a Davis decomposition*. In that sense the following embedding provides an alternative to Hardy-martingale-decomposition.

**Theorem 4.2.** *Let  $\eta > 0$ , and  $1 \leq p < \infty$ . For any  $X$  valued Hardy martingale  $g = (g_k)$  there exists an  $X$  valued Hardy martingale  $G = (G_k)$ , an increasing sequence of integers*

$$m(0) < m(1) < \dots < m(n) < \dots$$

*and a non-negative adapted process  $(\beta_k)$  such that*

$$\mathbb{E}(\sup_{k \in \mathbb{N}} \beta_k) \leq \sup_{k \in \mathbb{N}} \mathbb{E}(\|g_k\|_X), \quad (4.8)$$

*and so that the following conditions hold:*

1. *Small and previsible increments,*

$$\|\Delta G_k\|_X \leq \eta \beta_{k-1}, \quad (4.9)$$

2. *Almost identical  $L^p$  norms,*

$$(1 - \eta)\mathbb{E}(\|g_k\|_X^p) \leq \mathbb{E}(\|G_{m(k)}\|_X^p) \leq (1 + \eta)\mathbb{E}(\|g_k\|_X^p). \quad (4.10)$$

*and*

$$(1 - \eta)\mathbb{E}(\|\Delta g_k\|_X^p) \leq \mathbb{E}(\|G_{m(k)} - G_{m(k-1)}\|_X^p) \leq (1 + \eta)\mathbb{E}(\|\Delta g_k\|_X^p). \quad (4.11)$$

PROOF. The proof iterates Maurey's Theorem 4.1. First of all we may assume that the martingale  $g = (g_k)$  is finite, and that moreover each  $g_k$  is a trigonometric polynomial. To keep the notation simple we restrict the presentation to the case  $p = 1$ .

**Step 1 (Preparation).** Depending on the martingale  $(g_k)$  we select  $0 < \epsilon < \eta$  so that

$$\epsilon/\eta = (\sup \mathbb{E}(\|g_k\|_X)) / (\sum_{k \in \mathbb{N}} \mathbb{E}(\|\Delta g_k\|_X)). \quad (4.12)$$

Since  $g = (g_k)$  is finite we have in fact  $0 < \epsilon$ . Let  $\varphi = (\varphi_n)$  be the Hardy martingale (4.1) defined by  $\varphi_1(w) = \epsilon w_1$ , and

$$\varphi_n(w) = \varphi_{n-1}(w) + \epsilon(1 - |\varphi_{n-1}(w)|)^2 w_n, \quad w \in \mathbb{T}^{\mathbb{N}}.$$

We shorten the notation and put

$$\Omega = \mathbb{T}^{\mathbb{N}}.$$

**Step 2 (Substitution).** For  $k \in \mathbb{N}$  and  $u \in \Omega^k$  we write  $u = (u^{(1)}, \dots, u^{(k)})$  where  $u^{(1)} \in \Omega, \dots, u^{(k)} \in \Omega$ . Define

$$\Phi^k : \Omega^k \rightarrow \mathbb{T}^k, \quad \Phi^k(u) = (\varphi(u^{(1)}), \dots, \varphi(u^{(k)})), \quad (4.13)$$

and form the linear substitution operator

$$T : L^1(\mathbb{T}^k) \rightarrow L^1(\Omega^k), \quad Tf(u) = f(\Phi^k(u)). \quad (4.14)$$

Clearly,  $T$  is a contraction between the  $L^1$  spaces in (4.14).

Fix  $k \in \mathbb{N}$ ,  $v \in \Omega^{k-1}$ , and  $w \in \Omega$ . Then clearly  $u = (v, w) \in \Omega^k$  and we have

$$(Tg_k)(v, w) = g_k(\Phi^{k-1}(v), \varphi(w)).$$

We view  $g_{k-1}$  as a function on  $\mathbb{T}^k$  that does not depend on the last variable. Hence we may apply the substitution operator  $T$  to  $g_{k-1}$ , and since  $\mathbb{E}_{k-1}(g_k) = g_{k-1}$  we observe the following commutation relation between expectations

$$(Tg_{k-1})(v) = \mathbb{E}_{(w)}(Tg_k(v, w)). \quad (4.15)$$

**Step 3 (An intermediary Hardy martingale).** We fix  $v \in \Omega^{k-1}$  and form the Hardy martingale with respect to the last variable,

$$h_m(w) = g_k(\Phi^{k-1}(v), \varphi_m(w)), \quad m \in \mathbb{N}, w \in \Omega.$$

Theorem 4.1 asserts that  $h = (h_m)$  is an  $X$  valued Hardy martingale, and that its increments are small and uniformly bounded. Specifically, if we put

$$\alpha_{k-1}(v) = \mathbb{E}_{(w)}(\|Tg_k(v, w) - Tg_{k-1}(v)\|_X),$$

then

$$\sup_{w \in \Omega} \|\Delta h_n(w)\|_X \leq \epsilon \alpha_{k-1}(v)$$

and

$$\sup_{n \in \mathbb{N}} \mathbb{E}_{(w)} \|h_n\|_X = \mathbb{E}_{(w)} (\|Tg_k(v, w)\|_X),$$

where the integration is over  $w \in \Omega$



**Step 4 (Bounding the active variables).** Since  $g = (g_k)$  is assumed to be a finite martingale we pick now  $n \in \mathbb{N}$  so that

$$g = (g_k)_{k=1}^n \quad (4.16)$$

We approximate  $Tg_k$  by stopping the martingales  $\varphi = (\varphi_m)$  used in the definition of the linear substitutions  $T$ . We will replace the limit  $\varphi$  by one of its approximatants  $\varphi_m$ , thereby reduce the number of active variables.

For any  $m \in \mathbb{N}$  define the substitutions

$$T_m f(v) = f(\Phi_m^n(v)), \quad \Phi_m^n(v) = (\varphi_m(v^{(1)}), \dots, \varphi_m(v^{(n)})), \quad v \in \Omega^n,$$

Since  $n \in \mathbb{N}$  and  $\epsilon > 0$  are fixed, there exists  $K \in \mathbb{N}$  so that

$$\sup_{k \leq n} \mathbb{E}_{\Omega^n}(\|T_K(g_k) - T(g_k)\|_X) \leq \epsilon \sup_{k \leq n} \mathbb{E}_{\Omega}(\|g_k\|_X), \quad (4.17)$$

and

$$\sup_{k \leq n} \mathbb{E}_{\Omega^n}(\|T_K(\Delta g_k) - T(\Delta g_k)\|_X) \leq \epsilon \sup_{k \leq n} \mathbb{E}_{\Omega}(\|g_k\|_X), \quad (4.18)$$

where the integration on the left hand side is with respect to the normalized Haar measure of  $\Omega^n$ , and on the right hand side we integrate over  $\Omega = \mathbb{T}^{\mathbb{N}}$ .

By construction, for each  $k \leq n$ , the dependence of  $T_K(g_k)$  is only on the following variables,

$$(v_1^{(1)}, \dots, v_K^{(1)}, \dots, v_1^{(k-1)}, \dots, v_K^{(k-1)}, w_1, \dots, w_K).$$

Thus  $T_K(g_k)$  is un-ambiguously defined on the torus product

$$\mathbb{T}^{Kk} \subseteq \mathbb{T}^{Kn}.$$

**Step 5 (Conclusion).** Put  $N = Kn$ , where  $n \in \mathbb{N}$  respectively  $K \in \mathbb{N}$  are defined by (4.16) respectively (4.17). Finally we define the Hardy martingale  $G = (G_k)_{k=1}^N$  : Put

$$\rho : \Omega^n \rightarrow \mathbb{T}^{Kn}, \quad \rho(v) = (v_1^{(1)}, \dots, v_K^{(1)}, \dots, v_1^{(n)}, \dots, v_K^{(n)}),$$

then

$$G : \mathbb{T}^N \rightarrow X$$

is defined without ambiguity, by putting

$$G(z) = (T_K(g_n)(v)), \quad z = \rho(v).$$

Let  $m(k) = Kk$  then by the commutation relation (4.15)

$$G_{m(k)} = \mathbb{E}_{m(k)}(G)(z) = (T_K(g_k)(v)), \quad z = \rho(v),$$

and

$$(G_{m(k)} - G_{m(k-1)})(z) = T_K(g_k - g_{k-1})(v), \quad z = \rho(v).$$

Thus in view of (4.17) and (4.18) we verified (4.10) and (4.11).

Finally we let  $\mathbb{E}_K$  denote the conditional expectation projecting onto the first  $K$  variables of  $\Omega = \mathbb{T}^{\mathbb{N}}$ . Let

$$\mathbb{F}_K = \mathbb{E}_K \otimes \dots \otimes \mathbb{E}_K,$$

be the conditional expectation on  $\Omega^n$  given by the  $n$ -fold tensor product of  $\mathbb{E}_K$ . Theorem 4.1 asserts that for  $m(k-1) \leq j < m(k)$ , we have the pointwise estimate

$$\|\Delta G_j(z)\|_X \leq \epsilon \mathbb{F}_K(\alpha_{k-1}(v)), \quad z = \rho(v). \quad (4.19)$$

Define now the adapted process

$$\beta_{k-1}(z) = (\epsilon/\eta) \mathbb{F}_K(\alpha_{k-1}(v)), \quad z = \rho(v).$$

Clearly we have  $\mathbb{E}_{\Omega^n}(\sup_k \mathbb{F}_K(\alpha_{k-1})) \leq \mathbb{E}_{\Omega^n}(\sum \|T\Delta g_k\|_X)$ , and (4.12) –specifying the relation between  $\epsilon$  and  $\eta > 0$ –gives

$$\mathbb{E}(\sup_{k \in \mathbb{N}} \beta_k) \leq \sup_{k \in \mathbb{N}} \mathbb{E}(\|g_k\|_X). \quad (4.20)$$

Thus (4.19) translates to (4.9) and (4.20) gives (4.8). ■

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